Introduction to QCD at Colliders
Lecture II: Parton Branching and proton structure

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Slides available from http://theory.fnal.gov/people/ellis/Talks/fermi06/

Bibliography

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(Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology)
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Parton branching and proton structure

- Branching probabilities
- Parton Branching – kinematics
- Solution of massless Dirac equation and spinor products
- Parton Branching
- DGLAP equation
  - Quarks and gluons
  - Solution by moments
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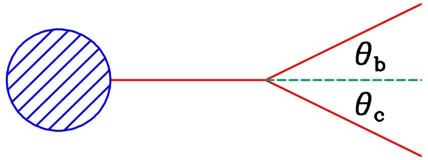
Branching probabilities



- We can write the matrix element squared for $n+1$ partons in the small-angle region in terms of that for n partons,

$$|\mathcal{M}_{n+1}|^2 \sim \frac{4g^2}{t} C\ F(z; h_a, h_b, h_c) + \mathrm{non-singular\ terms} |\mathcal{M}_n|^2$$

  where C is a colour factor functions $F(z)$ contain the momentum dependence of the branching probabilities.

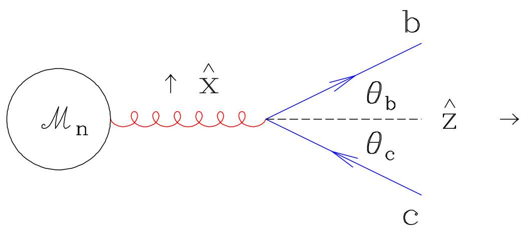
- After azimuthal averaging we obtain the splitting functions.

$$\sum_{h_a, h_b, h_c} \int \frac{d\phi}{2\pi} C\ F(z; h_a, h_b, h_c) = \hat{P}_{ba}(z);$$

  where $\hat{P}_{ba}(z)$ is the appropriate splitting function

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z) \ .$$

Parton branching - kinematics



$$
\begin{aligned}
p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\
p_b &= (E_b, +E_b \sin\theta_b, 0, +E_b \cos\theta_b) \\
p_c &= (E_c, -E_c \sin\theta_c, 0, +E_c \cos\theta_c)
\end{aligned}
$$

- the kinematics and notation for the branching of parton a into $b + c$. We assume that

$$p_b^2,\, p_c^2 \ll p_a^2 \equiv t$$

- a is an outgoing parton, which is called timelike branching since $t > 0$.
- The opening angle is $\theta = \theta_b + \theta_c$. Defining the energy fraction as

$$z = E_b/E_a = 1 - E_c/E_a \ ,$$

we have for small angles, $t = 2E_bE_c(1 - \cos\theta) = z(1-z)E_a^2\theta^2$

- using transverse momentum conservation,

$$\theta = \frac{1}{E_a}\sqrt{\frac{t}{z(1-z)}} = \frac{\theta_b}{1-z} = \frac{\theta_c}{z} \ .$$

- We consider first the case that all the partons a, b and c are gluons. There will be a factor in the amplitude proportional to $1/t$ from the propagator for gluon a, and a triple-gluon vertex factor

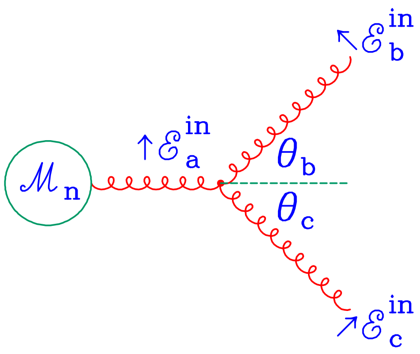
$$V_{ggg} = gf^{ABC}\varepsilon_a^\alpha \varepsilon_b^\beta \varepsilon_c^\gamma [g_{\alpha\beta}(p_a - p_b)_\gamma + g_{\beta\gamma}(p_b - p_c)_\alpha + g_{\gamma\alpha}(p_c - p_a)_\beta]$$

where ε_i^μ represents the polarization vector for gluon i. Note that here all momenta are defined as outgoing, so that $p_a = -p_b - p_c$. Using this and the conditions $\varepsilon_i \cdot p_i = 0$,

$$V_{ggg} = -2gf^{ABC}[(\varepsilon_a \cdot \varepsilon_b)(\varepsilon_c \cdot p_b) - (\varepsilon_b \cdot \varepsilon_c)(\varepsilon_a \cdot p_b) - (\varepsilon_c \cdot \varepsilon_a)(\varepsilon_b \cdot p_c)] \, .$$

- three gluons are almost on mass-shell, we can take their polarization vectors to be purely transverse. We shall resolve them into plane polarization states, $\varepsilon_i^{\text{in}}$ in the plane of branching and $\varepsilon_i^{\text{out}}$ normal to the plane

$$\begin{aligned} \varepsilon_i^{\text{in}} \cdot \varepsilon_j^{\text{in}} &= \varepsilon_i^{\text{out}} \cdot \varepsilon_j^{\text{out}} = -1 \\ \varepsilon_i^{\text{in}} \cdot \varepsilon_j^{\text{out}} &= \varepsilon_i^{\text{out}} \cdot p_j = 0 \, . \end{aligned}$$



- For small θ, neglecting terms of order θ^2, we have

$$\begin{aligned}
\varepsilon_a^{\text{in}} \cdot p_b &= -E_b\theta_b = -z(1-z)E_a\theta \\
\varepsilon_b^{\text{in}} \cdot p_c &= +E_c\theta = (1-z)E_a\theta \\
\varepsilon_c^{\text{in}} \cdot p_b &= -E_b\theta = -zE_a\theta \,.
\end{aligned}$$

- Note that every vertex factor is proportional to θ, which, together with the $1/t \propto 1/\theta^2$ from the propagator, gives a $1/\theta$ singularity in the amplitude.
- In the small-angle region

$$|\mathcal{M}_{n+1}|^2 \sim \frac{4g^2}{t} C_A F(z; \varepsilon_a, \varepsilon_b, \varepsilon_c) |\mathcal{M}_n|^2$$

where the colour factor $C_A = 3$ comes from $f^{ABC} f^{ABD} = C_A \delta^{CD}$ and the functions $F(z; \varepsilon_a, \varepsilon_b, \varepsilon_c)$

ε_a	ε_b	ε_c	$F(z; \varepsilon_a, \varepsilon_b, \varepsilon_c)$
in	in	in	$(1-z)/z + z/(1-z) + z(1-z)$
in	out	out	$z(1-z)$
out	in	out	$(1-z)/z$
out	out	in	$z/(1-z)$

Gluon splitting function

All polarization combinations not given in the table are forbidden. Defining $\langle F \rangle$ by averaging $F(z; \varepsilon_a, \varepsilon_b, \varepsilon_c)$ with respect to the polarization of a and summing over those of b and c, we find that

$$C_A \langle F \rangle \equiv \hat{P}_{gg}(z) = C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right]$$

where $\hat{P}_{gg}(z)$ is called the unregularized gluon splitting function

Soft enhancements and angular correlations

- the enhancements in the matrix elements at $z \to 0$ (b soft) and $z \to 1$ (c soft) are associated with the emission of a soft gluon polarized in the plane of branching.
- the correlation between the plane of branching and the polarization of the branching gluon. If we suppose that the polarization of gluon a is at an angle ϕ to the plane then the function F becomes

$$\begin{aligned}
F_\phi &\propto \sum_{\varepsilon_{b,c}} \left[\cos^2\phi \, |\mathcal{M}(\varepsilon_a^{\text{in}}, \varepsilon_b, \varepsilon_c)|^2 + \sin^2\phi \, |\mathcal{M}(\varepsilon_a^{\text{out}}, \varepsilon_b, \varepsilon_c)|^2 \right] \\
&= \cos^2\phi \left[\frac{1-z}{z} + \frac{z}{1-z} + 2z(1-z) \right] + \sin^2\phi \left[\frac{1-z}{z} + \frac{z}{1-z} \right] \\
&= \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) + z(1-z)\cos 2\phi \; .
\end{aligned}$$

- The first three terms give the unpolarized result and the last gives the correlation, which favours an orientation in which the polarization of the branching gluon lies in the plane of branching. The correlation is quite weak, however: its coefficient $z(1-z)$ vanishes in the enhanced regions $z \to 0, 1$ and reaches its maximum at $z = \frac{1}{2}$, where it is still only 1/9 of the unpolarized contribution.

Dirac eqn. for massless fermions

- The fermions involved in high energy processes can often be taken to be massless.
- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

  The Weyl representation is more suitable at high energy

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

  In the Weyl representation upper and lower components have different helicities.
- Both representations satisfy the same commutation relations.

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

- in the Weyl representation $\gamma^0 \gamma^i = \begin{pmatrix} \sigma^i & \mathbf{0} \\ \mathbf{0} & -\sigma^i \end{pmatrix}$. σ are the Pauli matrices.

- The Dirac eqns for the right- and left-handed components become,

$$\begin{pmatrix} p^- & -(p^1 - ip^2) \\ -(p^1 + ip^2) & p_+ \end{pmatrix} u_+(p) = 0, \quad \begin{pmatrix} p^+ & +(p^1 - ip^2) \\ +(p^1 + ip^2) & p_- \end{pmatrix} u_-(p) = 0,$$

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi_p} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-} e^{-i\varphi_p} \\ -\sqrt{p^+} \end{bmatrix},$$

where

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}}, \qquad p^\pm = p^0 \pm p^3.$$

In this representation the Dirac conjugate spinors are

$$\overline{u_+}(p) \equiv u_+^\dagger(p)\gamma^0 = \left[0, 0, \sqrt{p^+}, \sqrt{p^-} e^{-i\varphi_p}\right], \quad \overline{u_-}(p) = \left[\sqrt{p^-} e^{i\varphi_p}, -\sqrt{p^+}, 0, 0\right]$$

- Normalization

$$u_\pm^\dagger u_\pm = 2p^0$$

Interlude: spinor products

Introduce a bra and ket notation spinors corresponding to (massless) momenta p_i, $i = 1, 2, \ldots, n$ labelled by the index i

$$|i^{\pm}\rangle \equiv |p_i^{\pm}\rangle \equiv u_{\pm}(p_i) = v_{\mp}(p_i),$$
$$\langle i^{\pm}| \equiv \langle p_i^{\pm}| \equiv \overline{u_{\pm}}(p_i) = \overline{v_{\mp}}(p_i).$$

We define the basic spinor products by

$$\langle ij\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-}(p_i)u_+(p_j), \quad [ij] \equiv \langle i^+|j^-\rangle = \overline{u_+}(p_i)u_-(p_j).$$

The helicity projection implies that products like $\langle i^+|j^+\rangle$ vanish.

$$\langle i+|j+\rangle = \langle i-|j-\rangle = \langle ii\rangle = [ii] = 0$$

$$\langle ij\rangle = -\langle ji\rangle, \quad [ij] = -[ji]$$

We get explicit formulae for the spinor products valid for the case when both energies are positive, $p_i^0 > 0, \quad p_j^0 > 0$

$$\begin{aligned}
\langle i\, j\rangle &= \sqrt{p_i^- p_j^+}\, e^{i\varphi_{p_i}} - \sqrt{p_i^+ p_j^-}\, e^{i\varphi_{p_j}} = \sqrt{|s_{ij}|}\, e^{i\phi_{ij}}, \\
[i\, j] &= \sqrt{p_i^+ p_j^-}\, e^{-i\varphi_{p_j}} - \sqrt{p_i^- p_j^+}\, e^{-i\varphi_{p_i}} = -\sqrt{|s_{ij}|}\, e^{-i\phi_{ij}}
\end{aligned}$$

where $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$, and

$$\cos\phi_{ij} = \frac{p_i^1 p_j^+ - p_j^1 p_i^+}{\sqrt{|s_{ij}| p_i^+ p_j^+}}, \qquad \sin\phi_{ij} = \frac{p_i^2 p_j^+ - p_j^2 p_i^+}{\sqrt{|s_{ij}| p_i^+ p_j^+}}.$$

- The spinor products are, up to a phase, square roots of Lorentz products.
- The collinear limits of massless gauge amplitudes have square-root singularities; spinor products lead to very compact analytic representations of gauge amplitudes.

Branching probabilities

- Consider the case where

$$\begin{aligned} p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\ p_b &\sim (E_b, +E_b\theta_b, 0, +E_b) \\ p_c &\sim (E_c, -E_c\theta_c, 0, +E_c) \end{aligned}$$

Thus for example

$$u_+^\dagger(p_b) = \sqrt{2E_b}\left[1, \frac{\theta_b}{2}, 0, 0\right]$$

and

$$u_+(p_c) \equiv v_-(p_c) = \sqrt{2E_c}\begin{bmatrix} 1 \\ -\frac{\theta_c}{2} \\ 0 \\ 0 \end{bmatrix}$$

Hence for polarization vectors $\varepsilon_{in} = (0,1,0,0), \varepsilon_{out} = (0,0,1,0)$

$$g\bar{u}_+^{b\dagger}\,\gamma^0\gamma^1\,v_-^c = g\sqrt{4E_bE_c}\begin{pmatrix}1, & \frac{\theta_b}{2}\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 \\ -\frac{\theta_c}{2}\end{pmatrix} = -g\sqrt{E_bE_c}(\theta_b - \theta_c)$$

$$-g\bar{u}^b_+\gamma_\mu\varepsilon_a^{p\text{in}^\mu} v^c_- = g\sqrt{E_bE_c}(\theta_b-\theta_c) = g\sqrt{z(1-z)}(1-2z)E_a\theta\ ,$$

$$-g\bar{u}^b_+\gamma_\mu\varepsilon_a^{p\text{out}^\mu} v^c_- = ig\sqrt{E_bE_c}(\theta_b+\theta_c) = ig\sqrt{z(1-z)}E_a\theta\ ,$$

and the matrix element relation for the branching is

$$|\mathcal{M}_{n+1}|^2 \sim \frac{g^2}{t} T_R F(z;\varepsilon_a,\lambda_b,\lambda_c)|\mathcal{M}_n|^2$$

where the colour factor is now Tr$(t^At^A)/8 = T_R = 1/2$. The non-vanishing functions $F(z;\varepsilon_a,\lambda_b,\lambda_c)$ for quark and antiquark helicities λ_b and λ_c are

ε_a	λ_b	λ_c	$F(z;\varepsilon_a,\lambda_b,\lambda_c)$
in	$\pm$	$\mp$	$(1-2z)^2$
out	$\pm$	$\mp$	1

Summing over the polarizations we get

$$2\Big[(1-2z)^2+1\Big] = 4(z^2+(1-z)^2).$$

There is a strong anticorrelation between the polarization and the plane

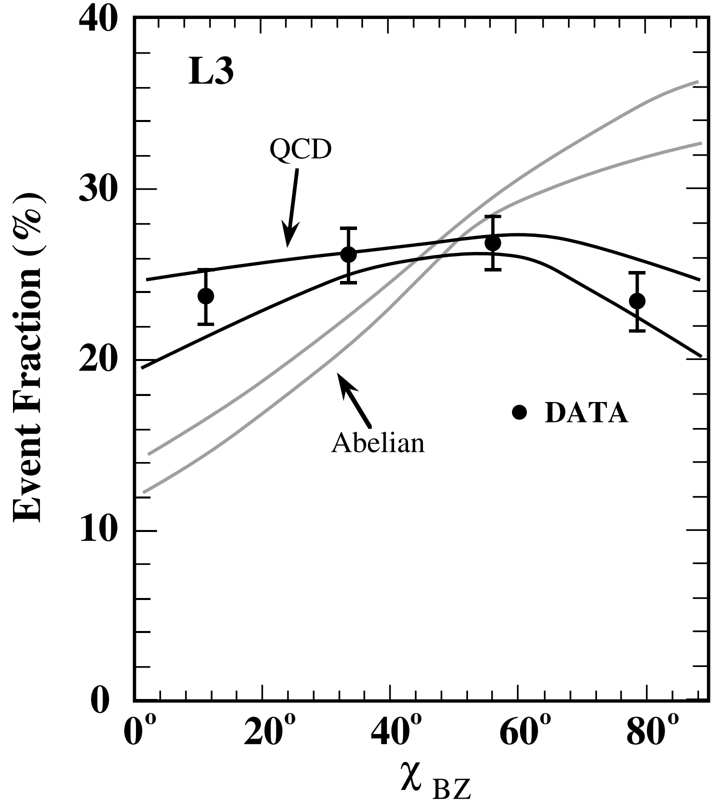
$$F_\phi \propto \cos^2\phi\,(1-2z)^2+\sin^2\phi = z^2+(1-z)^2-2z(1-z)\cos 2\phi\ .$$

Colour factors

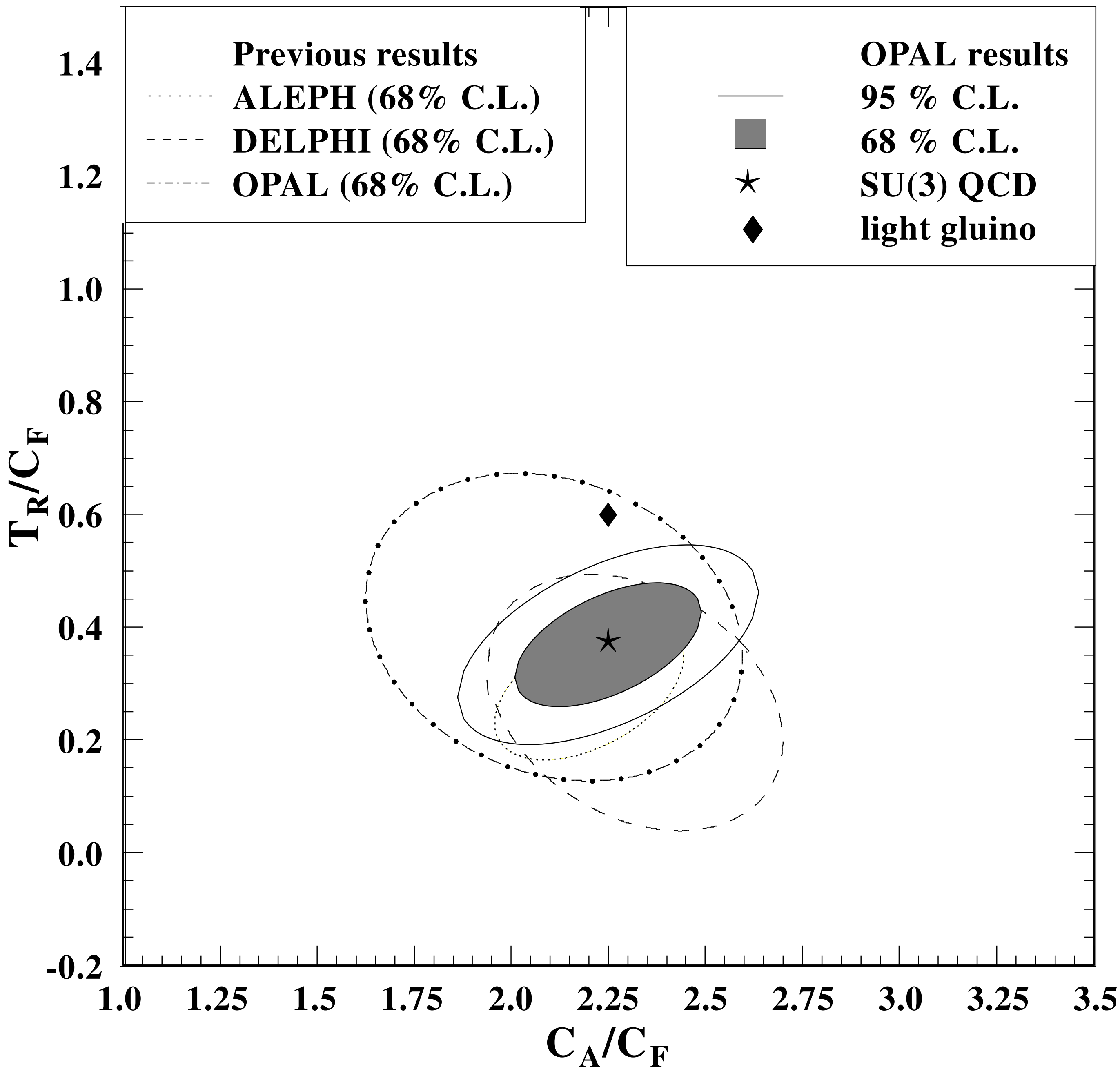
- The Bengtsson-Zerwas angle χ_{BZ}, defined as the angle between the planes defined by the two lowest and two highest energy jets:

$$\cos \chi_{\mathrm{BZ}} = \frac{(\boldsymbol{p}_1 \times \boldsymbol{p}_2) \cdot (\boldsymbol{p}_3 \times \boldsymbol{p}_4)}{|\boldsymbol{p}_1||\boldsymbol{p}_2||\boldsymbol{p}_3||\boldsymbol{p}_4|} \ .$$

- the figure shows the χ_{BZ} distribution measured by the L3 collaboration at LEP. The curves correspond to the Abelian and non-Abelian QCD theories, and the data exhibit a clear preference for the latter.



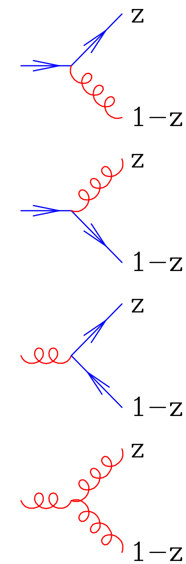
Colour factors II



Branching probabilities summary

- Including color factors we obtain the unregulated branching probabilities.

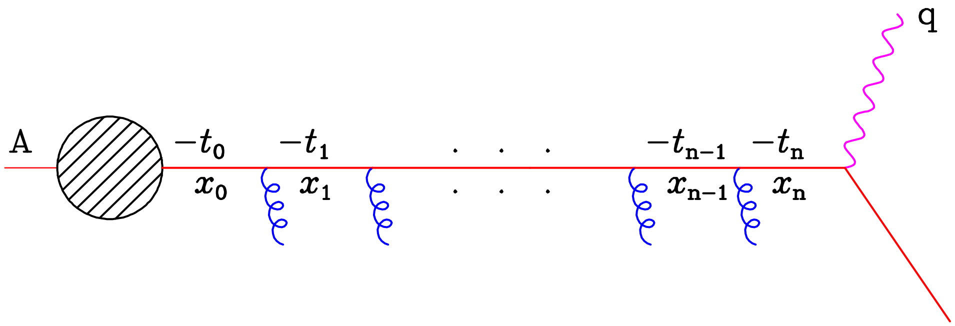
$$
\begin{aligned}
\hat{P}_{qq}(z) &= C_F \left[\frac{1+z^2}{(1-z)} \right], \\
\hat{P}_{gq}(z) &= C_F \left[\frac{1+(1-z)^2}{z} \right], \\
\hat{P}_{qg}(z) &= T_R \left[z^2 + (1-z)^2 \right], \\
\hat{P}_{gg}(z) &= C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z\,(1-z) \right].
\end{aligned}
$$



- $C_F = \frac{4}{3}, C_A = 3, T_R = \frac{1}{2}$.
- These are unregulated probabilities because they contain singularities at $z = 1$ and $z = 0$.

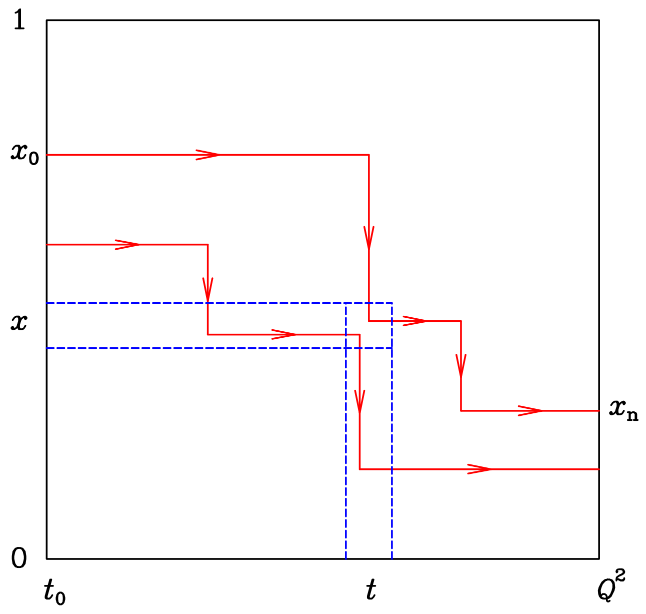
DGLAP equation

- Consider enhancement of higher-order contributions due to multiple small-angle parton emission, for example in deep inelastic scattering (DIS)



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.
- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful later for Monte Carlo simulation.



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.

Change in parton distribution

- Consider change in the parton distribution $D(x,t)$ when t is increased to $t+\delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.
- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned}
\delta D_{\text{in}}(x,t) &= \frac{\delta t}{t}\int_x^1 dx'\, dz\, \frac{\alpha_S}{2\pi}\hat{P}(z) D(x',t)\,\delta(x - zx') \\
&= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z) D(x/z,t)
\end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned}
\delta D_{\text{out}}(x,t) &= \frac{\delta t}{t} D(x,t)\int_0^x dx'\, dz\, \frac{\alpha_S}{2\pi}\hat{P}(z)\,\delta(x' - zx) \\
&= \frac{\delta t}{t} D(x,t)\int_0^1 dz\, \frac{\alpha_S}{2\pi}\hat{P}(z)
\end{aligned}$$

Change in parton distribution

- Change in population of element is

$$\begin{aligned}\delta D(x,t) &= \delta D_{\text{in}} - \delta D_{\text{out}} \\ &= \frac{\delta t}{t}\int_0^1 dz\,\frac{\alpha_S}{2\pi}\hat{P}(z)\left[\frac{1}{z}D(x/z,t) - D(x,t)\right] .\end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dx\; f(x)\, g(x)_+ = \int_0^1 dx\; [f(x) - f(1)]\, g(x) \; .$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+ \; ,$$

- Plus-prescription, like the Dirac-delta function, is only defined under integral sign.
- Plus-prescription includes some of the effects of virtual diagrams.

DGLAP

We obtain the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)D(x/z,t)\ .$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t.
- In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

Quarks and gluons

- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P_{ij}(z)D_j(x/z,t)\ .$$

- Quark ($i = q$) can enter element via either $q \to qg$ or $g \to q\bar{q}$, but can only leave via $q \to qg$. Thus plus-prescription applies only to $q \to qg$ part, giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2+(1-z)^2\right] \end{aligned}$$

- Gluon can arrive either from $g \to gg$ (2 contributions) or from $q \to qg$ (or $\bar{q} \to \bar{q}g$). Thus number arriving is

$$
\begin{aligned}
\delta D_{g,\text{in}} &= \frac{\delta t}{t} \int_0^1 dz \frac{\alpha_S}{2\pi} \left\{ \hat{P}_{gg}(z) \left[\frac{D_g(x/z,t)}{z} + \frac{D_g(x/(1-z),t)}{1-z} \right] \right. \\
&\quad \left. + \frac{\hat{P}_{qq}(z)}{1-z} \left[D_q\left(\frac{x}{1-z},t\right) + D_{\bar{q}}\left(\frac{x}{1-z},t\right) \right] \right\} \\
&= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} \left\{ 2\hat{P}_{gg}(z) D_g\left(\frac{x}{z},t\right) + \hat{P}_{qq}(1-z) \left[D_q\left(\frac{x}{z},t\right) + D_{\bar{q}}\left(\frac{x}{z},t\right) \right] \right\} ,
\end{aligned}
$$

- Gluon can leave by splitting into either gg or $q\bar{q}$, so that

$$
\delta D_{g,\text{out}} = \frac{\delta t}{t} D_g(x,t) \int_0^1 dz \frac{\alpha_S}{2\pi} \left[\hat{P}_{gg}(z) + N_f \hat{P}_{qg}(z)\, dz \right] .
$$

- After some manipulation we find

$$
\begin{aligned}
P_{gg}(z) &= 2C_A \left[\left(\frac{z}{1-z} + \frac{1}{2} z(1-z) \right)_+ + \frac{1-z}{z} + \frac{1}{2} z(1-z) \right] - \frac{2}{3} N_f T_R\, \delta(1-z) , \\
P_{gq}(z) &= P_{g\bar{q}}(z) = \hat{P}_{qq}(1-z) = C_F \frac{1+(1-z)^2}{z} .
\end{aligned}
$$

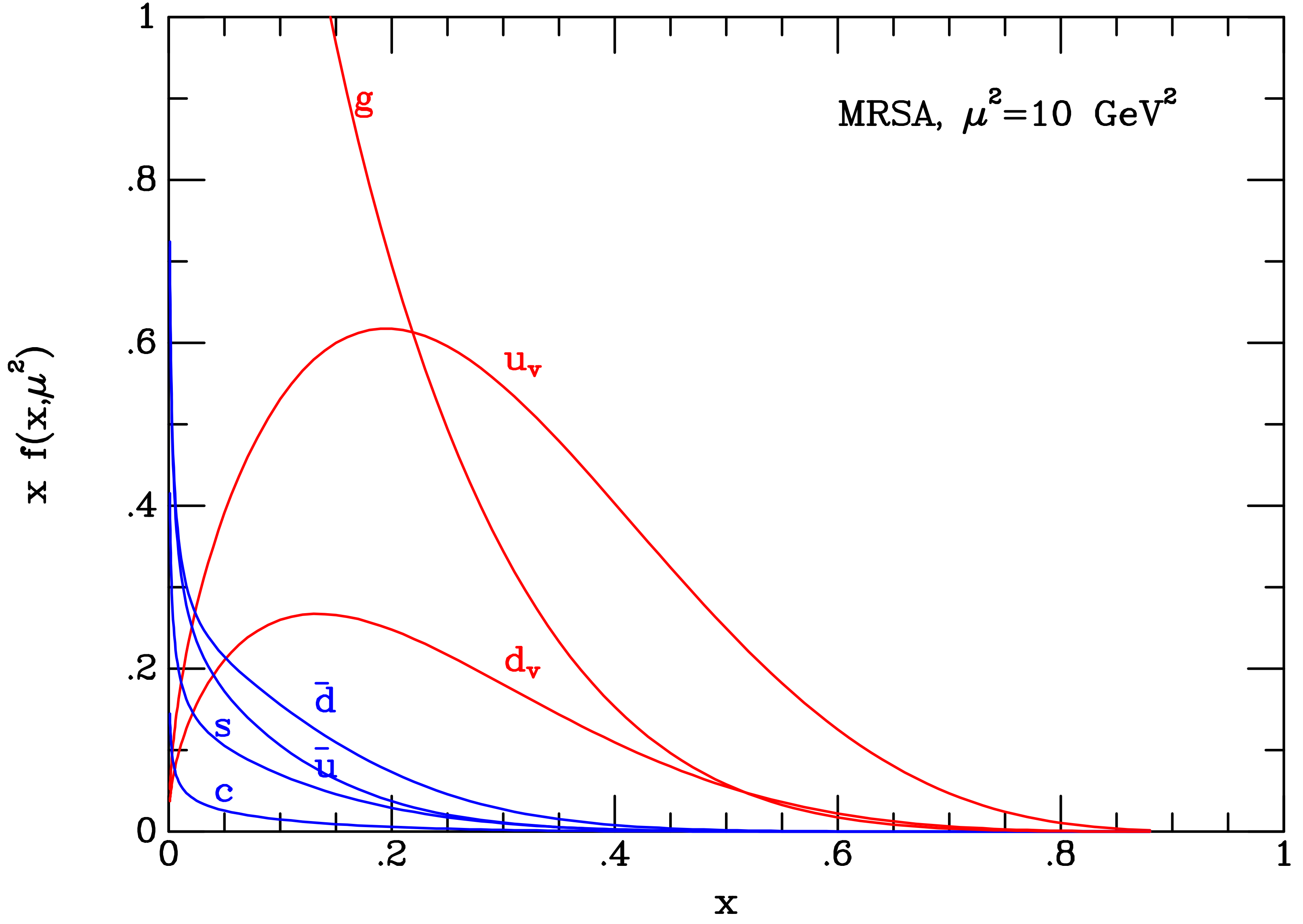
- Using definition of the plus-prescription, can check that

$$
\begin{aligned}
\left(\frac{z}{1-z} + \frac{1}{2}z(1-z)\right)_+ &= \frac{z}{(1-z)_+} + \frac{1}{2}z(1-z) + \frac{11}{12}\delta(1-z) \\
\left(\frac{1+z^2}{1-z}\right)_+ &= \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\ ,
\end{aligned}
$$

so P_{qq} and P_{gg} can be written in more common forms

$$
\begin{aligned}
P_{qq}(z) &= C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] \\
P_{gg}(z) &= 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \frac{1}{6}(11C_A - 4N_f T_R)\,\delta(1-z)\ .
\end{aligned}
$$

Parton distributions



Solution by moments

- Given $D_i(x,t)$ at some scale $t = t_0$, factorized structure of DGLAP equation means we can compute its form at any other scale.
- One strategy for doing this is to take moments (Mellin transforms) with respect to x:

$$\tilde{D}_i(N,t) = \int_0^1 dx\, x^{N-1}\, D_i(x,t)\ .$$

  Inverse Mellin transform is

$$D_i(x,t) = \frac{1}{2\pi i}\int_C dN\, x^{-N}\, \tilde{D}_i(N,t)\ ,$$

  where contour C is parallel to imaginary axis to right of all singularities of integrand.
- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t\frac{\partial}{\partial t}\tilde{D}_i(x,t) = \sum_j \gamma_{ij}(N,\alpha_S)\tilde{D}_j(N,t)$$

Anomalous dimensions

- The moments of splitting functions give PT expansion of anomalous dimensions γ_{ij}:

$$
\begin{aligned}
\gamma_{ij}(N,\alpha_S) &= \sum_{n=0}^{\infty} \gamma_{ij}^{(n)}(N) \left(\frac{\alpha_S}{2\pi}\right)^{n+1} \\
\gamma_{ij}^{(0)}(N) &= \tilde{P}_{ij}(N) = \int_0^1 dz\, z^{N-1}\, P_{ij}(z)
\end{aligned}
$$

- From above expressions for $P_{ij}(z)$ we find, ($\frac{1}{[1-z]_+} \to -\sum_{j=1}^{n-1} \frac{1}{j}$)

$$
\begin{aligned}
\gamma_{qq}^{(0)}(N) &= C_F\left[-\frac{1}{2} + \frac{1}{N(N+1)} - 2\sum_{k=2}^{N}\frac{1}{k}\right] \\
\gamma_{qg}^{(0)}(N) &= T_R\left[\frac{(2+N+N^2)}{N(N+1)(N+2)}\right] \\
\gamma_{gq}^{(0)}(N) &= C_F\left[\frac{(2+N+N^2)}{N(N^2-1)}\right] \\
\gamma_{gg}^{(0)}(N) &= 2C_A\left[-\frac{1}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - \sum_{k=2}^{N}\frac{1}{k}\right] - \frac{2}{3}N_f T_R\,.
\end{aligned}
$$

Scaling violation

- Consider combination of parton distributions which is flavour non-singlet, e.g. $D_V = D_{q_i} - D_{\bar{q}_i}$ or $D_{q_i} - D_{q_j}$. Then mixing with the flavour-singlet gluons drops out and solution for fixed α_S is

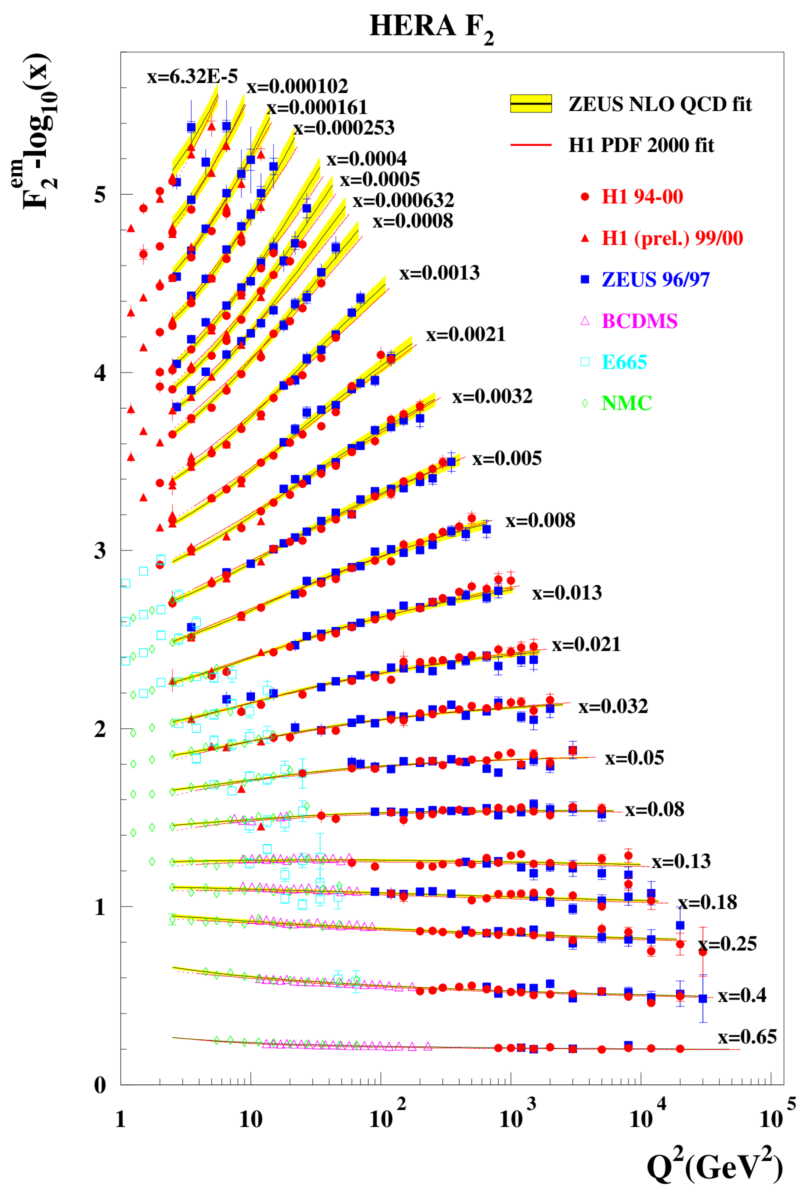
$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{t}{t_0}\right)^{\gamma_{qq}(N,\alpha_S)} ,$$

- We see that dimensionless function D_V, instead of being scale-independent function of x as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale t (hence the name anomalous dimensions).
- For running coupling $\alpha_S(t)$, scaling violation is power-behaved in $\ln t$ rather than t. Using leading-order formula $\alpha_S(t) = 1/b\ln(t/\Lambda^2)$, we find

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d_{qq}(N)}$$

  where $d_{qq}(N) = \gamma^{(0)}_{qq}(N)/2\pi b$.
- Flavour-singlet distribution and quantitative predictions will be discussed later.

Combined data on F2 proton



- Now $d_{qq}(1) = 0$ and $d_{qq}(N) < 0$ for $N \geq 2$. Thus as t increases V *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data:

Flavour singlet combination

- For flavour-singlet combination, define

$$\Sigma = \sum_i (q_i + \bar{q}_i) \; .$$

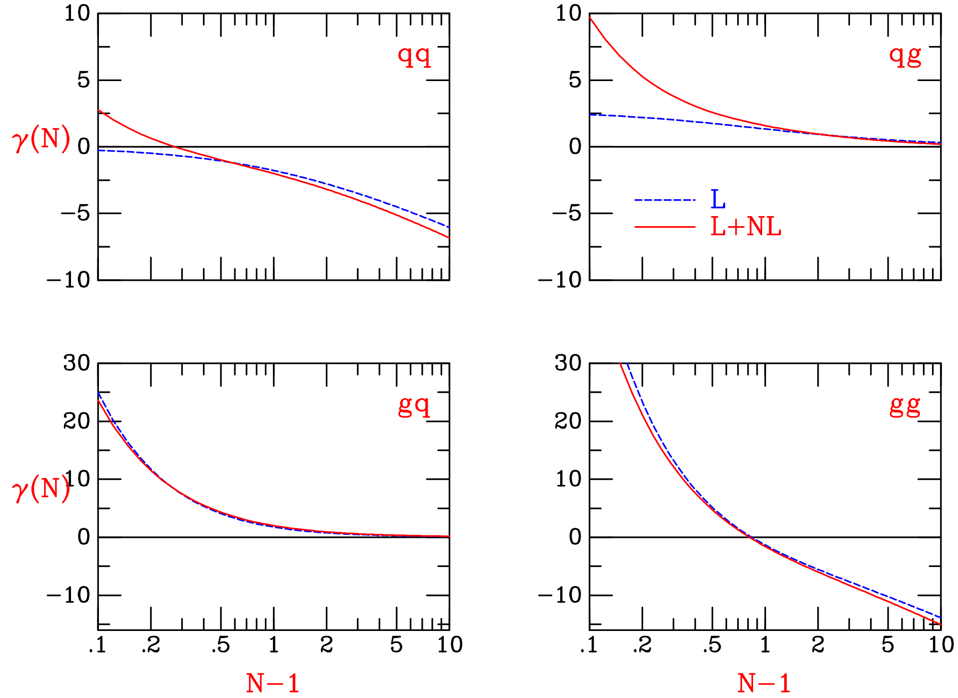
Then we obtain

$$\begin{aligned} t\frac{\partial \Sigma}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{qq}\otimes\Sigma + 2N_f P_{qg}\otimes g\right] \\ t\frac{\partial g}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{gq}\otimes\Sigma + P_{gg}\otimes g\right] \; . \end{aligned}$$

- Thus flavour-singlet quark distribution Σ mixes with gluon distribution g: evolution equation for moments has matrix form

$$t\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{g} \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & 2N_f\gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}\begin{pmatrix} \tilde{\Sigma} \\ \tilde{g} \end{pmatrix}$$

Anomalous dimension matrix as a function of N.



- Rapid growth at small N in gq and gg elements at lowest order
- $\ln N$ behaviour at large N in qq and gg elements
- NNLO now known
- Singlet anomalous dimension matrix has two real eigenvalues $\gamma_\pm$ given by

$$\gamma_\pm = \frac{1}{2}[\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg} - \gamma_{qq})^2 + 8N_f\gamma_{gq}\gamma_{qg}}] \,.$$

Solution of lowest order DGLAP matrix equation

The reduced DGLAP equation can be written as

$$\frac{d}{du}\left(\begin{array}{c}\tilde{\Sigma}(u)\\ \tilde{g}(u)\end{array}\right)=\mathbf{P}\left(\begin{array}{c}\tilde{\Sigma}(u)\\ \tilde{g}(u)\end{array}\right)$$

where $u=\frac{1}{2\pi b}\ln\frac{\alpha_S(\mu_0^2)}{\alpha_S(\mu^2)}$

- Define projection operators, $\mathbf{M}_\pm$

$$\mathbf{M}_+=\frac{1}{\gamma_+-\gamma_-}\Big[+\mathbf{P}-\gamma_-\mathbf{1}\Big],\;\;\mathbf{M}_-=\frac{1}{\gamma_+-\gamma_-}\Big[-\mathbf{P}+\gamma_+\mathbf{1}\Big],$$

  where $\mathbf{M}_\pm\mathbf{M}_\pm=\mathbf{M}_\pm,\mathbf{M}_+\mathbf{M}_-=\mathbf{M}_-\mathbf{M}_+=\mathbf{0}$, $\mathbf{M}_++\mathbf{M}_-=\mathbf{1}$ and

$$\mathbf{P}=\gamma_+\mathbf{M}_++\gamma_-\mathbf{M}_-$$

- The solution is

$$\left(\begin{array}{c}\tilde{\Sigma}(u)\\ \tilde{g}(u)\end{array}\right)=\Big[\mathbf{M}_+\exp(\gamma_+u)+\mathbf{M}_-\exp(\gamma_-u)\Big]\left(\begin{array}{c}\tilde{\Sigma}(0)\\ \tilde{g}(0)\end{array}\right)$$

Momentum partition vs Q^2

- For second moment

$$
\begin{aligned}
O^{+}(2,t) &= \Sigma(2,t) + g(2,t) \quad \text{with eigenvalue } 0\ , \\
O^{-}(2,t) &= \Sigma(2,t) - \frac{n_f}{4C_F} g(2,t) \quad \text{with eigenvalue } -\left(\frac{4}{3}C_F + \frac{n_f}{3}\right)\ .
\end{aligned}
$$

O^{+}, corresponds to the total momentum carried by the quarks and gluons, is independent of t. The eigenvector O^{-} vanishes in the limit $t \to \infty$:

$$
O^{-}(2,t) = \left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d^{-}(2)} \to 0, \quad \text{with} \quad d^{-}(2) = \frac{\gamma_{-}(2)}{2\pi b} = -\frac{\left(\frac{4}{3}C_F + \frac{1}{3}n_f\right)}{2\pi b}\ ,
$$

so that asymptotically we have

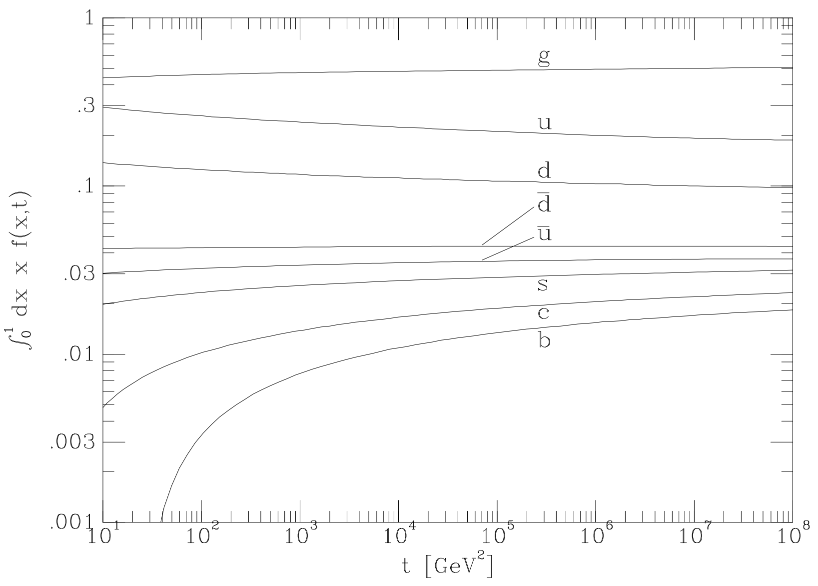
$$
\frac{\Sigma(2,t)}{g(2,t)} \to \frac{n_f}{4C_F} = \frac{3}{16} n_f\ .
$$

Asymptotia is approached slowly

The momentum fractions f_q and f_g in the $\mu^2 = t \to \infty$ limit are therefore

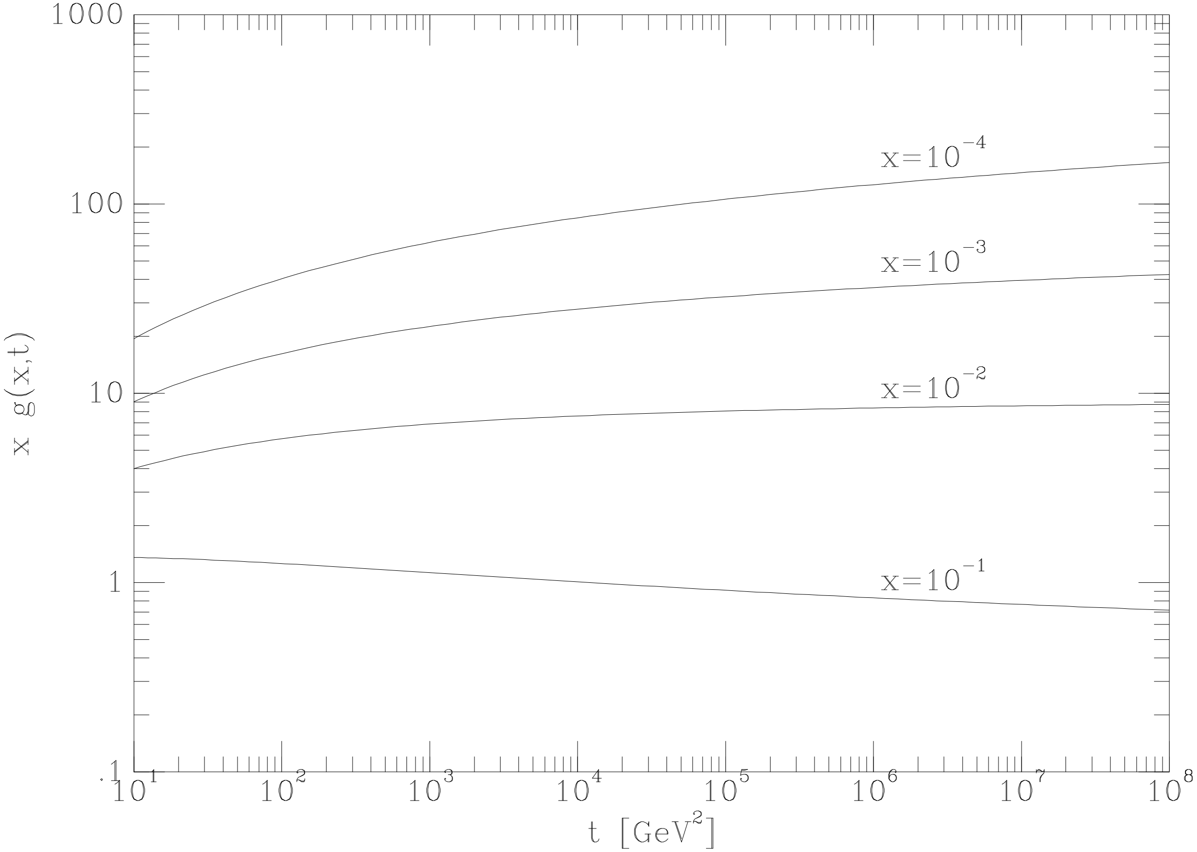
$$f_q = \frac{3n_f}{16 + 3n_f} \ , \quad f_g = \frac{16}{16 + 3n_f} \ .$$

- Scaling violation depends logarithmically on Q^2.
- Large variation at low Q^2



Gluon distribution

- Large number of gluons per unit rapidity
- The LHC is a copious source of gluons



Growth of gluon at small x

Using the moments of the DGLAP equation we have that,

$$t\frac{\partial}{\partial t}g(N,t)=\frac{\alpha(t)}{2\pi}\gamma_{gg}^{(0)}(N)g(N,t)$$

where the leading behaviour of the anomalous dimension is,

$$\gamma_{gg}^{(0)}(N)\approx\frac{2N_c}{N-1}\ .$$

In this limit the solution for the moments of the gluon distribution is

$$g(N,t)=g(N,t_0)\exp\left(\frac{\xi}{N-1}\right),$$

where ξ is defined by

$$\xi=\frac{N_c}{\pi}\int_{t_0}^{t}\frac{dt'}{t'}\alpha_S(t')=\frac{N_c}{\pi b}\ln\left(\frac{\ln t/\Lambda^2}{\ln t_0/\Lambda^2}\right)\ .$$

Inverse Mellin transform

To return to x space we perform the inverse Mellin transform

$$xg(x,t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dN\; x^{-(N-1)} g(N,t) \equiv \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dN\; g(N,t_0)\exp\Big[f(N)\Big]\;,$$

where $\mathrm{Re}C$ is to the right of all the singularities of $g(N,t)$ and the exponent f is

$$f(N) = (N-1)Y + \frac{\xi}{N-1}\;,$$

with $Y \equiv \ln(1/x)$. In the limit in which both Y and ξ become asymptotically large, we can estimate this integral by expanding about the saddle point of the exponential,

$$f(N) = f(N_0) + \frac{1}{2}f''(N_0)(N-N_0)^2, \qquad N_0 = 1+\sqrt{\frac{\xi}{Y}}\;,$$

which gives, for the asymptotic solution,

$$xg(x,t) \sim g(N_0,t_0)\exp(2\sqrt{\xi Y})\;.$$

$$g(x,\mu^2) \sim \frac{1}{x}\exp\sqrt{\frac{4N_c}{\pi b}\ln\frac{\ln\mu^2/\Lambda^2}{\ln\mu_0^2/\Lambda^2}\ln\frac{1}{x}},\quad N_c=3,\quad b=\frac{(33-2n_f)}{12\pi}\;.$$

Recap

- QCD at high momentum transferred is perturbative because of the property of asymptotic freedom.
- In the small-angle approximation, parton evolution can be represented as a branching process from higher values of x
- DGLAP equation predicts growth at small x and shrinkage at large x with increasing Q^2.
- In particular, the gluon grows rapidly at small x.